VI. *On the pear-shaped Figure of Equilibrium of a Rotating Mass of Liquid.*

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INTRODUCTION.

THIS is the sequel to a previous paper on "Ellipsoidal Harmonic Analysis" ('Phil. Trans..,' A, vol. 197, pp. 461–557). I here make use of the methods of that paper, and for brevity shall refer to it as "Harmonics."

The sections 1 to 4 are preparatory, and might have been included in "Harmonics," but seem more appropriate here. Section 5 is an independent investigation of so much of M. POINCARÉ's celebrated memoir on rotating liquid* as relates to the immediate object in view.

It is not necessary to say more here, since I give a short summary in the last section.

§ 1. *The Harmonics of the Third Degree.*

It was remarked in § 7 of "Harmonics" that this group of harmonics may be determined in rigorous algebraic form; I now proceed to do this.

The requisite formulæ are contained in § 6 of "Harmonics," and I take the several harmonics in succession. Throughout this section we have, of course, $i = 3$.

$s = 0$; type OEC, and $\mathfrak{P}_3(\nu) = q_0 P_3(\nu) + \beta q_2 P_3^2(\nu)$, with $q_0 = 1$.

The equation for $\beta\sigma$ is

$$\beta\sigma = \frac{\frac{1}{2}\beta^2\{3,1\}\{3,2\}}{4 + \beta\sigma} = \frac{15\beta^2}{1 + \frac{1}{4}\beta\sigma}.$$

If we write

$$(B_1)^2 = 1 + 15\beta^2,$$

the proper solution of the quadratic equation is

$$\beta\sigma = 2(B_1 - 1).$$

* 'Acta Mathematica,' vol. 7, 1885.

Also

$$\frac{q_2}{q_0} = \frac{1}{4+\beta\sigma} = \frac{1}{2(B_1+1)}, \text{ with } q_0 = 1.$$

Then since

$$P_3(\nu) = \tfrac{5}{2}\nu^3 - \tfrac{3}{2}\nu, \qquad P_3{}^2 = 15\nu(\nu^2-1),$$

we find

$$\mathfrak{P}_3(\nu) = \frac{1}{2\beta}(B_1 - 1 + 5\beta)\,\nu\left(\nu^2 - \frac{4-B_1}{5(1-\beta)}\right). \quad . \quad . \quad . \quad . \qquad (1).$$

$s=1$; type OOC, and $\mathsf{P}_3{}^1(\nu) = \sqrt{\dfrac{\nu^2 - \frac{1+\beta}{1-\beta}}{\nu^2-1}}\,(q_1'P_3{}^1(\nu) + \beta q_3'P_3{}^3(\nu))$, with $q_1' = 1$.

The equation for $\beta\sigma$ is

$$\beta\sigma + \tfrac{1}{2}\beta\,.\,3\,.\,4 = \frac{\frac{1}{4}\beta^2\{3,2\}\{3,3\}}{8+\beta\sigma},$$

or

$$\beta\sigma + 6\beta = \frac{\frac{15}{4}\beta^2}{2+\frac{1}{4}\beta\sigma}.$$

If we write

$$(B_2)^2 = 1 - \tfrac{3}{2}\beta(1-\beta),$$

the proper solution of the quadratic equation is

$$\beta\sigma = 4(B_2 - 1 - \tfrac{3}{4}\beta).$$

Also

$$\frac{2q_3}{q_1} = \frac{1}{8+\beta\sigma} = \frac{1}{4(B_2+1-\frac{3}{4}\beta)}, \quad \text{and} \quad \frac{q_3'}{q_1'} = 3\frac{q_3}{q_1}, \quad \text{with } q_1' = 1.$$

Then since $P_3{}^1(\nu) = \frac{3}{2}(5\nu^2-1)(\nu^2-1)^{\frac{1}{2}}$, $P_3{}^3(\nu) = 15(\nu^2-1)(\nu^2-1)^{\frac{1}{2}}$, we find

$$\mathsf{P}_3{}^1(\nu) = \frac{6}{\beta}(B_2 - 1 + 2\beta)\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\left(\nu^2 - \frac{3-\beta-2B_2}{5(1-\beta)}\right) \quad . \quad . \quad . \quad (2).$$

$s=1$, type OOS and $\mathfrak{P}_3{}^1(\nu) = q_1P_3{}^1(\nu) + \beta q_3P_3{}^3(\nu)$, with $q_1 = 1$.

The equation for $\beta\sigma$ is

$$\beta\sigma - \tfrac{1}{2}\beta\,.\,3\,.\,4 = \frac{\frac{1}{4}\beta^2\{3,2\}\{3,3\}}{8+\beta\sigma}.$$

A comparison with the last case shows that we have only to change the sign of β; accordingly if

$$(B_3)^2 = 1 + \tfrac{3}{2}\beta(1+\beta),$$

we have

$$\beta\sigma = 4(B_3 - 1 + \tfrac{3}{4}\beta),$$

$$\frac{2q_3}{q_1} = \frac{1}{4(B_3+1+\frac{3}{4}\beta)}, \text{ with } q_1 = 1.$$

On substitution we find

$$\mathfrak{P}_3^{\,1}(\nu) = \frac{2}{\beta}(B_3 - 1 + 3\beta)(\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{3 + \beta - 2B_3}{5(1-\beta)}\right) \quad . \quad . \quad . \quad . \quad (3).$$

$s = 2$; type OEC, $\mathfrak{P}_3^{\,2}(\nu) = \beta q_0 P_3(\nu) + q_2 P_3^{\,2}(\nu)$, with $q_2 = 1$.
The equation for $\beta\sigma$ is

$$\beta\sigma = \frac{-\frac{1}{2}\beta^2\{3, 1\}\{3, 2\}}{4 - \beta\sigma} = \frac{-15\beta^2}{1 - \frac{1}{4}\beta\sigma}.$$

We have already defined $(B_1)^2$ as $1 + 15\beta^2$, and find the proper solution of the quadratic equation to be

$$\beta\sigma = -2(B_1 - 1).$$

Also

$$\frac{2q_0}{q_2} = \frac{-\{3, 1\}\{3, 2\}}{4 - \beta\sigma} = \frac{-4(B_1 - 1)}{\beta^2}, \text{ with } q_2 = 1.$$

With the known values of P_3 and of $P_3^{\,2}$, we find

$$\mathfrak{P}_3^{\,2}(\nu) = \frac{5}{\beta}(1 + 3\beta - B_1)\,\nu\left(\nu^2 - \frac{4 + B_1}{5(1-\beta)}\right) \quad . \quad . \quad . \quad . \quad . \quad (4).$$

A comparison with (1) for $s = 0$ shows that the last factors in each only differ in the sign of B_1.

$s = 2$; type OES, $\mathsf{P}_3^{\,2}(\nu) = \sqrt{\dfrac{\nu^2 - \frac{1+\beta}{1-\beta}}{\nu^2 - 1}} \,.\, P_3^{\,2}(\nu)$.
Since $P_3^{\,2}(\nu) = 15\nu(\nu^2 - 1)$, we have at once

$$\mathsf{P}_3^{\,2}(\nu) = 15\nu(\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

$s = 3$; type OOC, $\mathsf{P}_3^{\,3}(\nu) = \sqrt{\dfrac{\nu^2 - \frac{1+\beta}{1-\beta}}{\nu^2 - 1}}\,[\beta q_1' P_3^{\,1}(\nu) + q_3' P_3^{\,3}(\nu)]$, with $q_3' = 1$.

The equation for $\beta\sigma$ is

$$\beta\sigma = \frac{-\frac{1}{4}\beta^2\{3, 2\}\{3, 3\}}{8 - \beta\sigma - 6\beta} = \frac{-15\beta^2}{8 - \beta\sigma - 6\beta}.$$

We have already defined

$$(B_2)^2 = 1 - \tfrac{3}{2}\beta(1 - \beta),$$

and find for the proper solution

$$\beta\sigma = 4(1 - \tfrac{3}{4}\beta - B_2).$$

Also

$$\frac{2q_1}{q_3} = \frac{-\{3, 2\}\{3, 3\}}{8 - \beta\sigma - 6\beta} = -\frac{16}{\beta^2}(B_2 - 1 + \tfrac{3}{4}\beta),$$

and

$$\frac{q_1'}{q_3'} = \frac{q_1}{3q_3}, \text{ with } q_3' = 1.$$

Whence on substitution

$$\mathsf{P}_3^{\,3}(\nu) = \frac{20}{\beta}(1 - B_2)\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\left(\nu^2 - \frac{3 - \beta + 2B_2}{5(1-\beta)}\right) \quad . \quad . \quad . \qquad (6).$$

$s = 3$; OOS, $\mathfrak{P}_3^{\,3}(\nu) = \beta q_1 P_3^{\,1}(\nu) + q_3 P_3^{\,3}(\nu)$, with $q_3 = 1$.
The equation for $\beta\sigma$ is

$$\beta\sigma = \frac{-\frac{1}{4}\beta^2\{3, 2\}\{3, 3\}}{8 - \beta\sigma + 6\beta}.$$

We may derive the result from the last case by introducing

$$(B_3)^2 = 1 + \tfrac{3}{2}\beta(1 + \beta),$$

and changing the sign of β, so that

$$\beta\sigma = 4(1 + \tfrac{3}{4}\beta - B_3),$$

$$\frac{2q_1}{q_3} = -\frac{16}{\beta^2}(B_3 - 1 - \tfrac{3}{4}\beta), \text{ with } q_3 = 1.$$

Whence on substitution

$$\mathfrak{P}_3^{\,3}(\nu) = \frac{60}{\beta}(1 + \beta - B_3)(\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{3 + \beta + 2B_3}{5(1-\beta)}\right) \quad . \quad . \quad . \quad . \qquad (7).$$

The forms of the corresponding functions of μ are the same, except that $(1 - \mu^2)^{\frac{1}{2}}$ and $\left(\frac{1+\beta}{1-\beta} - \mu^2\right)^{\frac{1}{2}}$ replace the corresponding factors.

I have not determined the cosine- and sine-functions, because they may be written down at once from the results already obtained. The three roots of the fundamental cubic are ν^2, μ^2, and $\frac{1-\beta\cos 2\phi}{1-\beta}$. Hence we have only to replace ν^2 by this last function in the seven formulæ (1)—(7) in order to obtain functions *proportional* to the seven cosine- and sine-functions. If the definition of the latter functions is to agree with that given in "Harmonics," the factors must be determined appropriately, but the question as to the value of the factor will not arise here.

§ 2. *Change of Notation.*

It will be convenient, with a view to future work, to change the notation, and I desire to adopt a notation which shall not only agree in the main with that used in "Harmonics," but shall also facilitate reference to a previous paper on JACOBI's ellipsoid ('Roy. Soc. Proc.,' vol. 41, pp. 319–336).

I write

$$\kappa^2 = \frac{1-\beta}{1+\beta}, \qquad \kappa'^2 = 1 - \kappa^2.$$

It may be noted that what I here write κ was denoted by κ' in "Harmonics," and *vice versâ*.

I have in general written the current co-ordinates ν, μ, ϕ, and the ellipsoid of reference ν_0, so that the squares of the semi-axes are

$$k^2\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right), \quad k^2(\nu_0^2 - 1), \quad k^2\nu_0^2.$$

I now propose to write as the squares of three semi-axes of the ellipsoid of reference

$$c^2\cos^2\gamma, \quad c^2(1-\kappa^2\sin^2\gamma), \quad c^2.$$

Comparing these two we see that

$$k = c\kappa\sin\gamma, \text{ and } \nu_0 = \frac{1}{\kappa\sin\gamma}.$$

For the current co-ordinates I retain ϕ and write

$$\nu = \frac{1}{\kappa\sin\psi}, \qquad \mu = \sin\theta.$$

The three roots of the fundamental cubic are therefore

$$\nu^2 = \frac{1}{\kappa^2\sin^2\psi}, \quad \mu^2 = \sin^2\theta, \quad \frac{1-\beta\cos 2\phi}{1-\beta} = \frac{1}{\kappa^2}(1-\kappa'^2\cos^2\phi).$$

The rectangular co-ordinates x, y, z are therefore now expressible as follows :—

$$\left.\begin{aligned} x &= \frac{c\sin\gamma}{\sin\psi}\,.\cos\psi\,(1-\kappa^2\sin^2\theta)^{\frac{1}{2}}\cos\phi, \\ y &= \frac{c\sin\gamma}{\sin\psi}\,.(1-\kappa^2\sin^2\psi)^{\frac{1}{2}}\cos\theta\sin\phi, \\ z &= \frac{c\sin\gamma}{\sin\psi}\,.\sin\theta\,(1-\kappa'^2\cos^2\phi)^{\frac{1}{2}} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (8).$$

These give

$$\frac{x^2}{\cos^2\psi} + \frac{y^2}{1 - \kappa^2\sin^2\psi} + z^2 = \frac{c^2\sin^2\gamma}{\sin^2\psi}.$$

At the surface $\psi = \gamma$, and we have

$$\frac{x^2}{\cos^2\gamma} + \frac{y^2}{1 - \kappa^2\sin^2\gamma} + z^2 = c^2.$$

In the formulæ for the third harmonics, in every case but one, and in two out of the five harmonics of the second degree, there occurs a factor of the form (ν^2 — constant); in each such case I write that constant in the form q^2/κ^2, and $q'^2 = 1 - q^2$. Thus q will have a different value for each harmonic.

It has been already remarked that for most purposes it is immaterial by what constants the several functions are multiplied. Although it would be easy to determine the constant in each case so as to make the function agree with its value as defined in "Harmonics," yet I shall not take that course, and shall omit factors as being in most cases redundant.

For the sake of completeness I will give the first and second harmonics in the new notation, as well as the third.

Since the harmonics of the first degree are expressed by

$$\mathfrak{P}_1(\nu) = \nu, \qquad \mathsf{P}_1{}^1(\nu) = \left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \qquad \mathfrak{P}_1{}^1(\nu) = (\nu^2 - 1)^{\frac{1}{2}},$$

it is clear that in the new notation

$$\left.\begin{array}{lll} \mathfrak{P}_1(\nu) = \dfrac{1}{\sin\psi}, & \mathsf{P}_1{}^1(\nu) = \cot\psi, & \mathfrak{P}_1{}^1(\nu) = \dfrac{(1-\kappa^2\sin^2\psi)^{\frac{1}{2}}}{\sin\psi}, \\ \mathfrak{P}_1(\mu) = \sin\theta, & \mathsf{P}_1{}^1(\mu) = (1-\kappa^2\sin^2\theta)^{\frac{1}{2}}, & \mathfrak{P}_1{}^1(\mu) = \cos\theta, \\ \mathsf{C}_1(\phi) = (1-\kappa'^2\cos^2\phi)^{\frac{1}{2}}, & \mathfrak{C}_1{}^1(\phi) = \cos\phi, & \mathfrak{S}_1{}^1(\phi) = \sin\phi \end{array}\right\} \quad (9).$$

It appears from § 12 of "Harmonics" that

$$\mathfrak{P}_2(\nu) = \nu^2 + \frac{\gamma}{\alpha}, \qquad \mathfrak{P}_2{}^2(\nu) = \nu^2 + \frac{\gamma'}{\alpha'},$$

where $\dfrac{\gamma}{\alpha} = \dfrac{B-2}{3(1-\beta)}, \quad \dfrac{\gamma'}{\alpha'} = -\dfrac{B+2}{3(1-\beta)}, \quad$ and $\quad B^2 = 1 + 3\beta^2.$

In accordance with the notation suggested above, let

$$\frac{q^2}{\kappa^2} = \frac{2 \mp B}{3(1-\beta)}.$$

Then substituting $\frac{1-\kappa^2}{1+\kappa^2}$ for β, we find

$$q^2 = \tfrac{1}{3}[1 + \kappa^2 \mp (1 - \kappa^2\kappa'^2)^{\frac{1}{2}}],$$

and for both cases

$$\kappa^2 = q^2 \frac{2 - 3q^2}{1 - 2q^2}.$$

Hence

$$\left.\begin{aligned} \mathfrak{P}_2(\nu) \text{ and } \mathfrak{P}_2^2(\nu) &= \frac{1 - q^2\sin^2\psi}{\sin^2\psi}, \\ \mathfrak{P}_2(\mu) \text{ and } \mathfrak{P}_2^2(\mu) &= -\left(1 - \frac{\kappa^2}{q^2}\sin^2\theta\right), \\ \mathfrak{C}_2(\phi) \text{ and } \mathfrak{C}_2^2(\phi) &= 1 - \frac{\kappa'^2}{q'^2}\cos^2\phi \end{aligned}\right\} \quad \ldots\ldots (10),$$

where $\kappa^2 = q^2 \frac{2-3q^2}{1-2q^2}$, and $q^2 = \frac{1}{3}[1 + \kappa^2 \mp (1 - \kappa^2\kappa'^2)^{\frac{1}{2}}]$, with upper sign for the first and the lower sign for the second.

It appears from (19) and (20), § 7, of "Harmonics" that

$$\left.\begin{aligned} \mathsf{P}_2^1(\nu) &= \frac{\cos\psi}{\sin^2\psi}, \\ \mathsf{P}_2^1(\mu) &= \sin\theta(1 - \kappa^2\sin^2\theta)^{\frac{1}{2}}, \\ \mathsf{C}_2^1(\phi) &= \cos\phi(1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}} \end{aligned}\right\} \quad \ldots\ldots (11),$$

and from (21) and (22) that

$$\left.\begin{aligned} \mathfrak{P}_2^1(\nu) &= \frac{(1 - \kappa^2\sin^2\psi)^{\frac{1}{2}}}{\sin^2\psi}, \\ \mathfrak{P}_2^1(\mu) &= \sin\theta\cos\theta, \\ \mathsf{S}_2^1(\phi) &= \sin\phi(1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}} \end{aligned}\right\} \quad \ldots\ldots (12).$$

Lastly, from (25) and (26)

$$\left.\begin{aligned} \mathsf{P}_2^2(\nu) &= \frac{\cos\psi(1 - \kappa^2\sin^2\psi)^{\frac{1}{2}}}{\sin^2\psi}, \\ \mathsf{P}_2^2(\mu) &= \cos\theta(1 - \kappa^2\sin^2\theta)^{\frac{1}{2}}, \\ \mathfrak{S}_2^2(\phi) &= \sin\phi\cos\phi \end{aligned}\right\} \quad \ldots\ldots (13).$$

Turning to the harmonics of the third degree, we found that in the two cases where the type is OEC,

$$\mathfrak{P}_3(\nu) \text{ and } \mathfrak{P}_3^2(\nu) = \nu\left(\nu^2 - \frac{4 \mp B_1}{5(1-\beta)}\right).$$

If we put $$\frac{q^2}{\kappa^2} = \frac{4 \mp B_1}{5(1-\beta)}, \text{ we find}$$

$$q^2 = \tfrac{2}{5}\left[1 + \kappa^2 \mp (1 - \tfrac{7}{4}\kappa^2 + \kappa^4)^{\frac{1}{2}}\right],$$

and $$\kappa^2 = q^2 \cdot \frac{4 - 5q^2}{3 - 4q^2}.$$

Therefore, with the above alternative form for q^2,

$$\left.\begin{aligned}
\mathfrak{P}_3(\nu) \text{ and } \mathfrak{P}_3^{\,2}(\nu) &= \frac{1 - q^2\sin^2\psi}{\sin^3\psi}, \\
\mathfrak{P}_3(\mu) \text{ and } \mathfrak{P}_3^{\,2}(\mu) &= -\sin\theta\left(1 - \frac{\kappa^2}{q^2}\sin^2\theta\right), \\
\mathsf{C}_3(\phi) \text{ and } \mathsf{C}_3^{\,2}(\phi) &= \left(1 - \frac{\kappa'^2}{q'^2}\cos^2\phi\right)(1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (14).$$

Again in the two cases where the type is OOC we found

$$\mathsf{P}_3^{\,1}(\nu) \text{ and } \mathsf{P}_3^{\,3}(\nu) = \left(\nu^2 - \frac{3 - \beta \mp 2B_2}{5(1-\beta)}\right)\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}.$$

Putting $$\frac{q^2}{\kappa^2} = \frac{3 - \beta \mp 2B_2}{5(1-\beta)},$$

we find $$q^2 = \tfrac{1}{5}\left(1 + 2\kappa^2 \mp (1 - \kappa^2 + 4\kappa^4)^{\frac{1}{2}}\right),$$

and $$\kappa^2 = q^2\,\frac{2 - 5q^2}{1 - 4q^2}.$$

Therefore, with the above alternative form for q^2,

$$\left.\begin{aligned}
\mathsf{P}_3^{\,1}(\nu) \text{ and } \mathsf{P}_3^{\,3}(\mu) &= \frac{\cos\psi\,(1 - q^2\sin^2\psi)}{\sin^3\psi}, \\
\mathsf{P}_3^{\,1}(\mu) \text{ and } \mathsf{P}_3^{\,3}(\mu) &= -(1 - \kappa^2\sin^2\theta)^{\frac{1}{2}}\left(1 - \frac{\kappa^2}{q^2}\sin^2\theta\right), \\
\mathfrak{C}_3^{\,1}(\phi) \text{ and } \mathfrak{C}_3^{\,3}(\phi) &= \cos\phi\left(1 - \frac{\kappa'^2}{q'^2}\cos^2\phi\right)
\end{aligned}\right\} \quad . \quad . \quad . \quad (15).$$

In the two cases where the type is OOS we found

$$\mathfrak{P}_3^{\,1}(\nu) \text{ and } \mathfrak{P}_3^{\,3}(\nu) = \left(\nu^2 - \frac{3 + \beta \mp 2B_3}{5(1-\beta)}\right)(\nu^2 - 1)^{\frac{1}{2}}.$$

Putting $$\frac{q^2}{\kappa^2} = \frac{3 + \beta \mp 2B_3}{5(1-\beta)},$$

we find $$q^2 = \tfrac{1}{5}(2 + \kappa^2 \mp (4 - \kappa^2\kappa'^2)^{\frac{1}{2}}),$$

and $$\kappa^2 = q^2 \frac{4 - 5q^2}{1 - 2q^2}.$$

Therefore, with the above alternative form for q^2,

$$\left.\begin{aligned} \mathfrak{P}_3^1(\nu) \text{ and } \mathfrak{P}_3^3(\nu) &= \frac{(1 - \kappa^2 \sin^2\psi)^{\frac{1}{2}}(1 - q^2 \sin^2\psi)}{\sin^3\psi}, \\ \mathfrak{P}_3^1(\mu) \text{ and } \mathfrak{P}_3^3(\mu) &= -\cos\theta\left(1 - \frac{\kappa^2}{q^2}\sin^2\theta\right), \\ \mathfrak{S}_3^1(\phi) \text{ and } \mathfrak{S}_3^3(\phi) &= \sin\phi\left(1 - \frac{\kappa'^2}{q'^2}\cos^2\phi\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (16).$$

The seventh of these harmonics, which is of type OES, stands by itself. We had

$$\mathsf{P}_3^2(\nu) = \nu(\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}.$$

This gives in the new notation

$$\left.\begin{aligned} \mathsf{P}_3^2(\nu) &= \frac{\cos\psi(1 - \kappa^2\sin^2\psi)^{\frac{1}{2}}}{\sin^3\psi}, \\ \mathsf{P}_3^2(\mu) &= \sin\theta\cos\theta(1 - \kappa^2\sin^2\theta)^{\frac{1}{2}}, \\ \mathsf{S}_3^2(\phi) &= \sin\phi\cos\phi(1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

The formulæ (9) to (17) give the fifteen sets of three functions constituting the fifteen harmonic functions of the first three degrees. It would be easy, although somewhat tedious, to find the coefficient by which each function is to be multiplied so that its definition may agree with that of the previous paper.

§ 3. *Expressions for the Solid Harmonics in Rectangular Co-ordinates.*

The three roots of the original cubic equation were ν^2, μ^2, $\dfrac{1 - \beta\cos 2\phi}{1 - \beta}$, and in the new notation the three roots of

$$\frac{x^2}{\omega^2 - 1/\kappa^2} + \frac{y^2}{\omega^2 - 1} + \frac{z^2}{\omega^2} = c^2\kappa^2\sin^2\gamma \quad \text{are} \quad \frac{1}{\kappa^2\sin^2\psi}, \ \sin^2\theta, \ \frac{1 - \kappa'^2\cos^2\phi}{\kappa^2}.$$

Hence it follows that we have the identity

$$\frac{x^2}{\omega^2 - 1/\kappa^2} + \frac{y^2}{\omega^2 - 1} + \frac{z^2}{\omega^2} - c^2\kappa^2 \sin^2\gamma = c^2\kappa^2 \sin^2\gamma \, \frac{\left(\frac{1}{\kappa^2 \sin^2\psi} - \omega^2\right)(\sin^2\theta - \omega^2)\left(\frac{1-\kappa'^2\cos^2\phi}{\kappa^2} - \omega^2\right)}{(1/\kappa^2 - \omega^2)(1 - \omega^2)\,\omega^2}.$$

Putting $\omega^2 = \dfrac{q^2}{\kappa^2}$,

$$\frac{x^2}{q'^2} + \frac{y^2}{\kappa^2 - q^2} - \frac{z^2}{q^2} + c^2\sin^2\gamma = \frac{c^2\sin^2\gamma}{(\kappa^2 - q^2)\sin^2\psi}(1 - q^2\sin^2\psi)\left(1 - \frac{\kappa^2}{q^2}\sin^2\theta\right)\left(1 - \frac{\kappa'^2}{q'^2}\cos^2\phi\right).$$

This expression, together with those for x, y, z in (8), enables us to write down the results at once. As before, I drop the several factors as being redundant for most purposes.

From (9)

$$\mathfrak{P}_1(\nu)\,\mathfrak{P}_1(\mu)\,\mathsf{C}_1(\phi) = z, \quad \mathsf{P}_1^1(\nu)\,\mathsf{P}_1^1(\mu)\,\mathfrak{C}_1^1(\phi) = x, \quad \mathfrak{P}_1^1(\nu)\,\mathfrak{P}_1^1(\mu)\,\mathfrak{S}_1^1(\phi) = y \quad . \quad . \quad . \quad (18).$$

From (10)

$$\mathfrak{P}_2(\nu)\,\mathfrak{P}_2(\mu)\,\mathfrak{C}_2(\phi) \text{ and } \mathfrak{P}_2^2(\nu)\,\mathfrak{P}_2^2(\mu)\,\mathfrak{C}_2^2(\phi) = q^2x^2 + \frac{q^2q'^2}{\kappa^2 - q^2}y^2 - q'^2z^2 + c^2q^2q'^2\sin^2\gamma \quad . \quad (19),$$

where $\quad q^2 = \frac{1}{3}[1 + \kappa^2 \mp (1 - \kappa^2\kappa'^2)^{\frac{1}{2}}]$, and $\kappa^2 = q^2\dfrac{2 - 3q^2}{1 - 2q^2}$,

so that $\quad \dfrac{q^2q'^2}{\kappa^2 - q^2} = 1 - 2q^2$.

From (11), (12), and (13)

$$\mathsf{P}_2^1(\nu)\,\mathsf{P}_2^1(\mu)\,\mathsf{C}_2^1(\phi) = xz, \quad \mathfrak{P}_2^1(\nu)\,\mathfrak{P}_2^1(\mu)\,\mathsf{S}_2^1(\phi) = yz, \quad \mathsf{P}_2^2(\nu)\,\mathsf{P}_2^2(\mu)\,\mathfrak{S}_2^2(\phi) = xy \quad . \quad . \quad . \quad (20).$$

From (14)

$$\mathfrak{P}_3(\nu)\,\mathfrak{P}_3(\mu)\,\mathsf{C}_3(\phi) \text{ and } \mathfrak{P}_3^2(\nu)\,\mathfrak{P}_3^2(\mu)\,\mathsf{C}_3^2(\phi) = z\left(q^2x^2 + \frac{q^2q'^2}{\kappa^2 - q^2}y^2 - q'^2z^2 + c^2q^2q'^2\sin^2\gamma\right) \quad . \quad (21),$$

where $\quad q^2 = \frac{2}{5}[1 + \kappa^2 \mp (1 - \frac{7}{4}\kappa^2 + \kappa^4)^{\frac{1}{2}}]$, and $\kappa^2 = q^2\dfrac{4 - 5q^2}{3 - 4q^2}$,

so that $\quad \dfrac{q^2q'^2}{\kappa^2 - q^2} = 3 - 4q^2$.

From (15)

$$\mathsf{P}_3^1(\nu)\,\mathsf{P}_3^1(\mu)\,\mathfrak{C}_3^1(\phi) \text{ and } \mathsf{P}_3^3(\nu)\,\mathsf{P}_3^3(\mu)\,\mathfrak{C}_3^3(\phi) = x\left(q^2x^2 + \frac{q^2q'^2}{\kappa^2 - q^2}y^2 - q'^2z^2 + c^2q^2q'^2\sin^2\gamma\right) \quad . \quad (22),$$

where $q^2 = \frac{1}{5}(1 + 2\kappa^2 \mp (1 - \kappa^2 + 4\kappa^4)^{\frac{1}{2}})$, and $\kappa^2 = q^2 \frac{2 - 5q^2}{1 - 4q^2}$,

so that
$$\frac{q^2 q'^2}{\kappa^2 - q^2} = 1 - 4q^2.$$

From (16)

$$\mathfrak{P}_3^1(\nu)\,\mathfrak{P}_3^1(\mu)\,\mathfrak{S}_3^1(\phi) \text{ and } \mathfrak{P}_3^3(\nu)\,\mathfrak{P}_3^3(\mu)\,\mathfrak{S}_3^3(\phi) = y\left(q^2x^2 + \frac{q^2q'^2}{\kappa^2 - q^2}y^2 - q'^2z^2 + c^2q^2q'^2\sin^2\gamma\right) \quad . \quad (23),$$

where $q^2 = \frac{1}{5}(2 + \kappa^2 \mp (4 - \kappa^2\kappa'^2)^{\frac{1}{2}})$ and $\kappa^2 = q^2\frac{4 - 5q^2}{1 - 2q^2}$,

so that
$$\frac{q^2q'^2}{\kappa^2 - q^2} = \tfrac{1}{3}(1 - 2q^2).$$

Lastly, from (17),

$$\mathsf{P}_3^2(\nu)\,\mathsf{P}_3^2(\mu)\,\mathsf{S}_3^2(\phi) = xyz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

It is easy to verify that each of these expressions satisfies LAPLACE'S equation.

§ 4. *The Expression for the Q-functions in Elliptic Integrals.*

In this paper I drop the factors $\mathfrak{E}$ and E which were found to be necessary when the Q-functions were expressed in series.

We make the following definition :—

$$\mathfrak{P}_i^s(\nu_0)\,\mathfrak{Q}_i^s(\nu_0) = [\mathfrak{P}_i^s(\nu_0)]^2 \int_{\nu_0}^{\infty} \frac{d\nu}{[\mathfrak{P}_i^s(\nu)^2](\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}},$$

and a similar formula holds for $\mathsf{P}_i^s\,\mathsf{Q}_i^s$.

It is clear that $\mathfrak{P}_i^s$ may be multiplied by any constant factor without changing the result; hence we may use the forms which have been found in §§ 2, 3.

The notation must now be changed.

We have $\nu = \dfrac{1}{\kappa\sin\psi}$ and $\nu_0 = \dfrac{1}{\kappa\sin\gamma}$. Therefore, when ψ is adopted as variable, the limits are γ to 0, and the sign of the whole is changed.

But
$$d\nu = -\frac{\cos\psi}{\kappa\sin^2\psi}\,d\psi,$$

and
$$(\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} = \frac{\cos\psi\,(1 - \kappa^2\sin^2\psi)^{\frac{1}{2}}}{\kappa^2\sin^2\psi}.$$

Therefore
$$\int_{\nu_0}^{\infty}\frac{d\nu}{(\nu^2 - 1)^{\frac{1}{2}}\,\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}} = \kappa\int_0^{\gamma}\frac{d\psi}{(1 - \kappa^2\sin^2\psi)^{\frac{1}{2}}}.$$

In accordance with the usage in elliptic integrals, I write

$$\Delta^2 = 1 - \kappa^2 \sin^2 \psi$$

under the integral sign, or $1 - \kappa^2 \sin^2 \gamma$ outside the integral.

I shall also for brevity write

$$\Delta_1^2 = 1 - q^2 \sin^2 \psi$$

under the integral, or $1 - q^2 \sin^2 \gamma$ outside the integral.

We have then

$$\mathfrak{P}_i^s(\nu_0)\,\mathfrak{Q}_i^s(\nu_0) = \kappa\,[\mathfrak{P}_i^s(\nu_0)]^2 \int_0^\gamma \frac{d\psi}{[\mathfrak{P}_i^s(\nu)]^2\,\Delta}.$$

I apply this formula successively to the several functions, as given in (9) to (17), and introduce the abridged notation just defined, but I do not reiterate the special meanings to be attached to the symbol q in each case.

Since $\mathfrak{P}_0(\nu) = 1$, we have (dropping the now unnecessary suffix 0),

$$\left.\begin{aligned}
&\mathfrak{P}_0(\nu)\,\mathfrak{Q}_0(\nu) = \kappa \int_0^\gamma \frac{d\psi}{\Delta}, \\
&\mathfrak{P}_1(\nu)\,\mathfrak{Q}_1(\nu) = \frac{\kappa}{\sin^2\gamma} \int_0^\gamma \frac{\sin^2\psi\, d\psi}{\Delta}, \\
&\mathsf{P}_1^1(\nu)\,\mathsf{Q}_1^1(\nu) = \kappa \cot^2\gamma \int_0^\gamma \frac{\tan^2\psi}{\Delta}\, d\psi, \\
&\mathfrak{P}_1^1(\nu)\,\mathfrak{Q}_1^1(\nu) = \frac{\kappa\,\Delta^2}{\sin^2\gamma} \int_0^\gamma \frac{\sin^2\psi}{\Delta^3}\, d\psi, \\
&\mathfrak{P}_2(\nu)\,\mathfrak{Q}_2(\nu) \text{ and } \mathfrak{P}_2^2(\nu)\,\mathfrak{Q}_2^2(\nu) = \frac{\kappa\,\Delta_1^4}{\sin^4\gamma} \int_0^\gamma \frac{\sin^4\psi}{\Delta_1^4\,\Delta}\, d\psi, \\
&\mathsf{P}_2^1(\nu)\,\mathsf{Q}_2^1(\nu) = \frac{\kappa\cos^2\gamma}{\sin^4\gamma} \int_0^\gamma \frac{\sin^4\psi}{\cos^2\psi\,\Delta}\, d\psi, \\
&\mathfrak{P}_2^1(\nu)\,\mathfrak{Q}_2^1(\nu) = \frac{\kappa\Delta^2}{\sin^4\gamma} \int_0^\gamma \frac{\sin^4\psi}{\Delta^3}\, d\psi, \\
&\mathsf{P}_2^3(\nu)\,\mathsf{Q}_2^2(\nu) = \frac{\kappa\cos^2\gamma\Delta^2}{\sin^4\gamma} \int_0^\gamma \frac{\sin^4\psi}{\cos^2\psi\Delta^3}\, d\psi, \\
&\mathfrak{P}_3(\nu)\,\mathfrak{Q}_3(\nu) \text{ and } \mathfrak{P}_3^2(\nu)\,\mathfrak{Q}_3^2(\nu) = \frac{\kappa\Delta_1^4}{\sin^6\gamma} \int_0^\gamma \frac{\sin^6\psi}{\Delta_1^4\Delta}\, d\psi, \\
&\mathsf{P}_3^1(\nu)\,\mathsf{Q}_3^1(\nu) \text{ and } \mathsf{P}_3^3(\nu)\,\mathsf{Q}_3^3(\nu) = \frac{\kappa\cos^2\gamma\Delta_1^4}{\sin^6\gamma} \int_0^\gamma \frac{\sin^6\psi}{\cos^2\psi\Delta_1^4\Delta}\, d\psi, \\
&\mathfrak{P}_3^1(\nu)\,\mathfrak{Q}_3^1(\nu) \text{ and } \mathfrak{P}_3^3(\nu)\,\mathfrak{Q}_3^3(\nu) = \frac{\kappa\Delta_1^4\Delta^2}{\sin^6\gamma} \int_0^\gamma \frac{\sin^6\psi}{\Delta_1^4\Delta^3}\, d\psi, \\
&\mathsf{P}_3^2(\nu)\,\mathfrak{Q}_3^2 = \frac{\kappa\cos^2\gamma\Delta^2}{\sin^6\gamma} \int_0^\gamma \frac{\sin^6\psi}{\cos^2\psi\Delta^3}\, d\psi
\end{aligned}\right\} \quad . \quad . \quad (25).$$

All these integrals are expressible in terms of the elliptic integrals

$$F = \int_0^\gamma \frac{d\psi}{\Delta}, \qquad E = \int_0^\gamma \Delta d\psi, \qquad \Pi = \int_0^\gamma \frac{d\psi}{\Delta_1^2 \Delta}.$$

It will, however, be found that in fact the coefficient of Π vanishes in every case. The cases of $i = 0$ and $i = 1$ are very simple, and we have

$$\mathfrak{P}_0 \mathfrak{Q}_0 = \kappa F,$$

$$\mathfrak{P}_1 \mathfrak{Q}_1 = \frac{\kappa}{\sin^2\gamma}\left(\frac{1}{\kappa^2} F - \frac{1}{\kappa^2} E\right),$$

$$\mathsf{P}_1{}^1 \mathsf{Q}_1{}^1 = \kappa \cot^2\gamma \left(\frac{1}{\kappa'^2} \Delta \tan\gamma - \frac{1}{\kappa'^2} E\right),$$

$$\mathfrak{P}_1{}^1 \mathfrak{Q}_1{}^1 = \frac{\kappa\Delta^2}{\sin^2\gamma}\left(-\frac{1}{\kappa^2} F + \frac{1}{\kappa^2\kappa'^2} E - \frac{\sin\gamma\cos\gamma}{\kappa'^2\Delta}\right).$$

It is possible by direct differentiation to verify the following results, although the verification will be found pretty tedious.

$$\int \frac{\sin^4\psi}{\Delta_1{}^4\Delta} d\psi = \frac{(2-3q^2)q^2 - \kappa^2(1-2q^2)}{2q^4q'^2(\kappa^2-q^2)} \Pi + \frac{2q'^2-1}{2q^4q'^2} F - \frac{1}{2q^2q'^2(\kappa^2-q^2)} E + \frac{\Delta \sin\psi\cos\psi}{2q'^2(\kappa^2-q^2)\Delta_1{}^2},$$

$$\int \frac{\sin^4\psi}{\cos^2\psi\Delta} d\psi = -\frac{1}{\kappa^2} F + \frac{1-2\kappa^2}{\kappa^2\kappa'^2} E + \frac{1}{\kappa'^2} \Delta \tan\psi,$$

$$\int \frac{\sin^4\psi}{\Delta^3} d\psi = -\frac{2}{\kappa^4} F + \frac{1+\kappa'^2}{\kappa^4\kappa'^2} E - \frac{\sin\psi\cos\psi}{\kappa^2\kappa'^2\Delta},$$

$$\int \frac{\sin^4\psi}{\cos^2\psi\Delta^3} d\psi = \frac{1}{\kappa^2\kappa'^2} F - \frac{1+\kappa^2}{\kappa^2\kappa'^4} E + \frac{\tan\psi}{\kappa'^4\Delta}\left[2 - (1+\kappa^2)\sin^2\psi\right].$$

These are all the integrals needed for the harmonics of the second degree. In the case of the first we have

$$\kappa^2 = q^2 \frac{2-3q^2}{1-2q^2}.$$

Thus the coefficient of Π vanishes and the results are

$$\mathfrak{P}_2(\nu)\,\mathfrak{Q}_2(\nu) \text{ and } \mathfrak{P}_2{}^2(\nu)\,\mathfrak{Q}_2{}^2(\nu) = \frac{\kappa\Delta_1{}^4}{\sin^4\gamma}\left[\frac{1-2q^2}{2q^4q'^2} F - \frac{1-2q^2}{2q^4q'^4} E + \frac{(1-2q^2)\Delta\sin\gamma\cos\gamma}{2q^2q'^4\Delta_1{}^2}\right],$$

$$\mathsf{P}_2{}^1(\nu)\,\mathsf{Q}_2{}^1(\nu) = \frac{\kappa\cos^2\gamma}{\sin^4\gamma}\left[-\frac{1}{\kappa^2} F + \frac{1-2\kappa^2}{\kappa^2\kappa'^2} E + \frac{1}{\kappa'^2}\Delta\tan\gamma\right],$$

$$\mathfrak{P}_2{}^1(\nu)\,\mathfrak{Q}_2{}^1(\nu) = \frac{\kappa\Delta^2}{\sin^4\gamma}\left[-\frac{2}{\kappa^4} F + \frac{1+\kappa'^2}{\kappa^4\kappa'^2} E - \frac{\sin\gamma\cos\gamma}{\kappa^2\kappa'^2\Delta}\right],$$

$$\mathsf{P}_2{}^2(\nu)\,\mathsf{Q}_2{}^2(\nu) = \frac{\kappa\cos^2\gamma\Delta^2}{\sin^4\gamma}\left[\frac{1}{\kappa^2\kappa'^2} F - \frac{1+\kappa^2}{\kappa^2\kappa'^4} E + \frac{\tan\gamma(2-(1+\kappa^2)\sin^2\gamma)}{\kappa'^4\Delta}\right].$$

In the first of these $q^2 = \frac{1}{3}[1 + \kappa^2 \mp (1 - \kappa^2\kappa'^2)^{\frac{1}{2}}]$ and $\kappa^2 = q^2 \dfrac{2-3q^2}{1-2q^2}$.

The following integrals may also be verified by differentiation :

$$\int \frac{\sin^6 \psi}{\Delta_1^4 \Delta} d\psi = \frac{q^2(4-5q^2) - \kappa^2(3-4q^2)}{2q^6 q'^2 (\kappa^2 - q^2)} \Pi + \frac{2q^2 q'^2 + \kappa^2(3-4q^2)}{2\kappa^2 q^6 q'^2} F - \frac{\kappa^2(3-2q^2) - 2q^2 q'^2}{2\kappa^2 q^4 q'^2 (\kappa^2 - q^2)} E + \frac{\Delta \sin\psi \cos\psi}{2q^2 q'^2 (\kappa^2 - q^2) \Delta_1^2} \quad . \quad (26),$$

$$\int \frac{\sin^6 \psi}{\cos^2 \psi \Delta_1^4 \Delta} d\psi = \frac{\kappa^2(1-4q^2) - q^2(2-5q^2)}{2q^4 q'^4 (\kappa^2 - q^2)} \Pi + \frac{4q^2 - 1}{2q^4 q'^4} F + \frac{1 + 2q^4 - \kappa^2(2q^2+1)}{2\kappa'^2 q^2 q'^4 (\kappa^2 - q^2)} E + \frac{\Delta \tan\psi}{\kappa'^2 q'^4} - \frac{\Delta \sin\psi \cos\psi}{2q'^4 (\kappa^2 - q^2) \Delta_1^2} \quad . \quad (27),$$

$$\int \frac{\sin^6 \psi}{\Delta_1^4 \Delta^3} d\psi = \frac{\kappa^2(1-2q^2) - q^2(4-5q^2)}{2q^4 q'^2 (\kappa^2 - q^2)^2} \Pi + \frac{2q^2 q'^2 - \kappa^2(1-2q^2)}{2\kappa^2 q^4 q'^2 (\kappa^2 - q^2)} F + \frac{2q^2 q'^2 + \kappa^2 \kappa'^2}{2\kappa^2 \kappa'^2 q^2 q'^2 (\kappa^2 - q^2)^2} E - \frac{\sin\psi \cos\psi}{\kappa'^2 (\kappa^2 - q^2)^2 \Delta} - \frac{\Delta \sin\psi \cos\psi}{2q'^2 (\kappa^2 - q^2)^2 \Delta_1^2} \quad . \quad (28),$$

$$\int \frac{\sin^6 \psi}{\cos^2 \psi \Delta^3} d\psi = \frac{2-\kappa^2}{\kappa^4 \kappa'^2} F - \frac{2(1-\kappa^2\kappa'^2)}{\kappa^4 \kappa'^4} E + \frac{\sin\psi \cos\psi}{\kappa^2 \kappa'^4 \Delta} + \frac{\Delta \tan\psi}{\kappa'^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

Now in (26) we have to put

$$\kappa^2 = q^2 \frac{4-5q^2}{3-4q^2};$$

in (27)

$$\kappa^2 = q^2 \frac{2-5q^2}{1-4q^2};$$

and in (28)

$$\kappa^2 = q^2 \frac{4-5q^2}{1-2q^2}.$$

Introducing these values, and taking the integrals between the limits γ and 0, we find :

$$\mathfrak{P}_3 \mathfrak{Q}_3 \text{ and } \mathfrak{P}_3^2 \mathfrak{Q}_3^2 = \frac{\kappa \Delta_1^4}{\sin^6 \gamma} \left\{ \frac{7q'^2 - 1}{2\kappa^2 q^4 q'^2} F - \frac{2q'^4 + 5q'^2 - 1}{2\kappa^2 q^4 q'^4} E + \frac{(4q'^2 - 1)}{2q^4 q'^4} \frac{\Delta \sin\gamma \cos\gamma}{\Delta_1^2} \right\} \quad . \quad . \quad . \quad . \quad (30).$$

$$\mathsf{P}_3^1 \mathsf{Q}_3^1 \text{ and } \mathsf{P}_3^3 \mathsf{Q}_3^3 = \frac{\kappa \cos^2\gamma \Delta_1^4}{\sin^6 \gamma} \left\{ \frac{4q^2 - 1}{2q^4 q'^4} F + \frac{1 - 5q^2 - 2q^4}{2\kappa'^2 q^4 q'^4} E - \left(\frac{1 - 7q^2 - (1 - 5q^2 - 2q^4) \sin^2\gamma}{2\kappa'^2 q^2 q'^4} \right) \frac{\Delta \tan\gamma}{\Delta_1^2} \right\} \quad . \quad (31).$$

$$\mathfrak{P}_3^1 \mathfrak{Q}_3^1 \text{ and } \mathfrak{P}_3^3 \mathfrak{Q}_3^3 = \frac{\kappa \Delta_1^4 \Delta^2}{\sin^6 \gamma} \left\{ - \frac{(1-2q^2)(2-3q^2)}{6\kappa^2 q^4 q'^4} F + \frac{2 - 11q^2 q'^2}{6\kappa^2 \kappa'^2 q^4 q'^4} E - \left(\frac{1 - 5q^2 + 6q^4 - q^2(2 - 11q^2 q'^2) \sin^2\gamma}{6\kappa'^2 q^4 q'^4} \right) \frac{\sin\gamma \cos\gamma}{\Delta \Delta_1^2} \right\} \quad . \quad (32).$$

$$\mathsf{P}_3^2\mathsf{Q}_3^2 = \frac{\kappa\cos^2\gamma\Delta^2}{\sin^6\gamma}\left\{\frac{1+\kappa'^2}{\kappa^4\kappa'^2}F - \frac{2(1-\kappa^2\kappa'^2)}{\kappa^4\kappa'^4}E + \left(\frac{1+\kappa^2-(1+\kappa^4)\sin^2\gamma}{\kappa^2\kappa'^4}\right)\frac{\tan\gamma}{\Delta}\right\} \quad . \quad (33).$$

In (30) $q^2 = \frac{2}{5}[1+\kappa^2 \mp (1-\frac{7}{4}\kappa^2+\kappa^4)^{\frac{1}{2}}]$, $\kappa^2 = q^2\dfrac{4-5q^2}{3-4q^2}$.

In (31) $q^2 = \frac{1}{5}[1+2\kappa^2 \mp (1-\kappa^2+4\kappa^4)^{\frac{1}{2}}]$, $\kappa^2 = q^2\dfrac{2-5q^2}{1-4q^2}$.

In (32) $q^2 = \frac{1}{5}[2+\kappa^2 \mp (4-\kappa^2\kappa'^2)^{\frac{1}{2}}]$, $\kappa^2 = q^2\dfrac{4-5q^2}{1-2q^2}$.

§ 5. *Bifurcation of Jacobi's Ellipsoid.*

If a mass of liquid be rotating like a rigid body about an axis, x, with uniform angular velocity ω, the determination of the figure of equilibrium may be treated as a statical problem, if the mass be subjected to a potential $\frac{1}{2}\omega^2(y^2+z^2)$.

The energy lost in the concentration of a body from a condition of infinite dispersion is equal to the potential of the body in its final configuration at the position of each molecule, multiplied by the mass of the molecule and summed throughout the body. In the proposed system, as rendered a statical one, it is necessary to add $\frac{1}{2}\omega^2(y^2+z^2)$ to the gravitation potential before making the summation. If A denotes the moment of inertia of the body about x, this latter portion of the sum is $\frac{1}{2}A\omega^2$, and is therefore the kinetic energy of the system.

If dm_1, dm_2 denote any pair of molecules and D_{12} the distance between them, and E the energy lost, we have

$$E = \frac{1}{2}\int\frac{dm_1\,dm_2}{D_{12}} + \frac{1}{2}A\omega^2.$$

If the system had been considered as a dynamical one, the expression for the energy of the system, say U, would have resembled that for E, but the former of these terms would have presented itself with a negative sign.

It is clear that the variation of $\frac{1}{2}A\omega^2$, when the moment of momentum is kept constant, is equal and opposite to the variation of the same function when the angular velocity is kept constant.

The condition for a figure of equilibrium is that U shall be stationary for constant moment of momentum, or E stationary for constant ω, in both cases subject to the condition of constancy of volume. The variations in question lead to identical results, and I shall proceed from the variation of E.

If
$$\Psi = \int_0^\infty \frac{du}{(u+a^2)^{\frac{1}{2}}(u+b^2)^{\frac{1}{2}}(u+c^2)^{\frac{1}{2}}},$$

the internal potential of an ellipsoid of mass M and semi-axes a, b, c is

$$\tfrac{3}{4}M\left[\Psi + \frac{x^2}{a}\frac{d\Psi}{da} + \frac{y^2}{b}\frac{d\Psi}{db} + \frac{z^2}{c}\frac{d\Psi}{dc}\right].$$

Hence
$$\tfrac{1}{2}\int \frac{dm_1 dm_2}{D_{12}} = \tfrac{3}{8}M\int_0^\infty \left[\Psi + \frac{x^2}{a}\frac{d\Psi}{da} + \ldots\right] dm.$$

Now if A, B, C denote the principal moments of inertia of the ellipsoid about x, y, z,

$$\int x^2 dm = \tfrac{1}{2}(C + B - A) = \tfrac{1}{5}Ma^2,$$

and similar formulæ hold for the two other axes.

Therefore

$$\tfrac{1}{2}\int \frac{dm_1 dm_2}{D_{12}} = \tfrac{3}{8}M^2\left[\Psi + \tfrac{1}{5}\left(a\frac{d\Psi}{da} + b\frac{d\Psi}{db} + c\frac{d\Psi}{dc}\right)\right].$$

But since Ψ is a homogeneous function of degree -1 in a, b, c, the sum of the three differential terms is equal to $-\Psi$. Hence this expression is equal to $\tfrac{3}{10}M^2\Psi$.

Since
$$\tfrac{1}{2}A\omega^2 = \tfrac{1}{10}M(b^2 + c^2)\omega^2,$$

we have
$$E = \tfrac{3}{10}M^2\left[\Psi + \frac{b^2 + c^2}{3M}\omega^2\right].$$

If E be varied, whilst abc and ω are constant, it is stationary if

$$\frac{d\Psi}{da}\delta a + \left(\frac{d\Psi}{db} + \frac{2b}{3M}\omega^2\right)\delta b + \left(\frac{d\Psi}{dc} + \frac{2c}{3M}\omega^2\right)\delta c = 0,$$

$$\frac{\delta a}{a} + \frac{\delta b}{b} + \frac{\delta c}{c} = 0.$$

Eliminating δa, δb, δc we have the well-known conditions for JACOBI's ellipsoid

$$\left.\begin{aligned} \frac{2\omega^2 b^2}{3M} &= a\frac{d\Psi}{da} - b\frac{d\Psi}{db}, \\ \frac{2\omega^2 c^2}{3M} &= a\frac{d\Psi}{da} - c\frac{d\Psi}{dc}, \\ \frac{1}{b^2}\left(a\frac{d\Psi}{da} - b\frac{d\Psi}{da}\right) &= \frac{1}{c^2}\left(a\frac{d\Psi}{da} - c\frac{d\Psi}{dc}\right). \end{aligned}\right\} \quad \ldots \ldots (34)$$

If we add together the first two of these, and avail ourselves of the property that Ψ is homogeneous of degree -1, we easily prove that the stationary value of E is

$$E = \tfrac{9}{20}M^2\left[\Psi + a\frac{d\Psi}{da}\right].$$

Since the potential of the ellipsoid must satisfy POISSON'S equation

$$\frac{d\Psi}{ada} + \frac{d\Psi}{bdb} + \frac{d\Psi}{cdc} = -\frac{2}{abc}.$$

Also

$$a\frac{d\Psi}{da} + b\frac{d\Psi}{dc} + c\frac{d\Psi}{dc} = -\Psi.$$

By means of these and two out of the three equations (34), we may eliminate the differentials of Ψ, and writing ρ for the density find

$$\frac{\omega^2}{2\pi\rho} = \frac{\Psi abc\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 6}{(b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

I do not happen to have seen this form for the angular velocity of JACOBI'S ellipsoid in any book.

It is easy also to show that the stationary value of E may be written

$$E = \tfrac{9}{20}M^2\frac{\left[(b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 4\right]\Psi - 2\frac{b^2 + c^2}{abc}}{(b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 6}.$$

We may now express the potential, say V, of the system entirely in terms of Ψ and $a\frac{d\Psi}{da}$, for

$$V = \tfrac{3}{4}M\left[\Psi + \frac{x^2}{a}\frac{d\Psi}{da} + \frac{y^2}{b^2}\left(a\frac{d\Psi}{da} - \frac{2\omega^2 b^2}{3M}\right) + \frac{z^2}{c^2}\left(a\frac{d\Psi}{da} - \frac{2\omega^2 c^2}{3M}\right)\right] + \tfrac{1}{2}\omega^2(y^2 + z^2),$$

$$= \tfrac{3}{4}M\left[\Psi + a\frac{d\Psi}{da}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\right].$$

We thus verify that V is constant over the surface of the ellipsoid.

Let g denote the value of gravity at the surface. Then if dn be an element of the outward normal, $g = -\frac{dV}{dn}$. Since

$$\frac{dx}{dn} = \frac{px}{a^2}, \quad \frac{dy}{dn} = \frac{py}{b^2}, \quad \frac{dz}{dn} = \frac{pz}{c^2},$$

where
$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4},$$

$$g = -\tfrac{3}{2} M a \frac{d\Psi}{da} p \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) = -\tfrac{3}{2} \frac{M}{p} a \frac{d\Psi}{da}.$$

Now change the notation and write

$$a^2 = k^2 \left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right), \quad b^2 = k^2 (\nu_0^2 - 1), \quad c^2 = k^2 \nu_0^2,$$

$$u = k^2 (\nu^2 - \nu_0^2).$$

Then
$$\Psi = \frac{2}{k} \int_{\nu_0}^{\infty} \frac{d\nu}{(\nu^2 - \frac{1+\beta}{1-\beta})^{\frac{1}{2}} (\nu^2 - 1)^{\frac{1}{2}}},$$

$$a \frac{d\Psi}{da} = -\frac{2}{k} \left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right) \int_{\nu_0}^{\infty} \frac{d\nu}{(\nu^2 - \frac{1+\beta}{1-\beta})^{\frac{3}{2}} (\nu^2 - 1)^{\frac{1}{2}}}.$$

Now
$$\mathfrak{P}_0(\nu) = 1, \quad \mathsf{P}_1^1(\nu) = \left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}},$$

and
$$\mathfrak{P}_i^s(\nu_0)\, \mathfrak{Q}_i^s(\nu_0) = [\mathfrak{P}_i^s(\nu_0)]^2 \int_{\nu_0}^{\infty} \frac{d\nu}{[\mathfrak{P}_i^s]^2 (\nu^2 - \frac{1+\beta}{1-\beta})^{\frac{1}{2}} (\nu^2 - 1)^{\frac{1}{2}}};$$

so that
$$\left.\begin{aligned} \Psi &= \frac{2}{k} \mathfrak{P}_0(\nu_0)\, \mathfrak{Q}_0(\nu_0), \\ a \frac{d\Psi}{da} &= -\frac{2}{k} \mathsf{P}_1^1(\nu_0)\, \mathsf{Q}_1^1(\nu_0), \\ \text{and} \quad g &= \frac{3M}{pk} \mathsf{P}_1^1(\nu_0)\, \mathsf{Q}_1^1(\nu_0) \end{aligned}\right\} \quad \ldots \ldots \ldots (36).$$

We may note in passing that the condition for a Jacobian ellipsoid (the last equation of (34)) is reducible to the form

$$\frac{\kappa \Delta^2}{\sin^4 \gamma} \int_0^{\gamma} \frac{\sin^4 \psi}{\Delta^3} d\psi = \kappa \cot^2 \gamma \int_0^{\gamma} \frac{\tan^2 \psi}{\Delta} d\psi.$$

On examining the series of functions given in (25), we see that it may be written

$$\mathfrak{P}_2^1(\nu_0)\, \mathfrak{Q}_2^1(\nu_0) = \mathsf{P}_1^1(\nu_0)\, \mathsf{Q}_1^1(\nu)_0.$$

This agrees with M. POINCARÉ's equation (1) on p. 341 of his memoir.

We will now suppose that the body, instead of being an ellipsoid, is an ellipsoidal harmonic deformation of an ellipsoid, which is itself a figure of equilibrium for rotation ω.

The addition to E will consist of three parts; first that due to the mutual

energy of the layer of deformation; secondly that due to the ellipsoid and the layer; thirdly that due to the change in the moment of inertia.

If a subscript l denotes integration throughout the space occupied by the layer, U the potential of the ellipsoid, and dv an element of volume,

$$\delta E = \tfrac{1}{2}\int_l \frac{dm_1 dm_2}{D_{12}} + \int_l U\rho dv + \tfrac{1}{2}\omega^2 \int_l (y^2 + z^2)\,\rho dv.$$

If ζ denotes the thickness of the layer standing on the element $d\sigma$, the first of these terms is $\tfrac{1}{2}\rho^2 \iint \frac{\zeta_1\zeta_2 d\sigma_1 d\sigma_2}{D_{12}}$.

The value of $U + \tfrac{1}{2}\omega^2(y^2+z^2)$ throughout the layer is equal to $V_0 - g\zeta'$, where V_0 is the constant value of $U + \tfrac{1}{2}\omega^2(y^2+z^2)$ over the surface of the ellipsoid, and ζ' is the distance measured along the normal to the element $d\zeta' d\sigma$ of volume.

Hence
$$\int_l U\rho dv + \tfrac{1}{2}\omega^2 \int_l (y^2+z^2)\,\rho dv = \iint_0^\zeta \rho\,(V_0 - g\zeta')\,d\zeta' d\sigma.$$

Since V_0 is constant and the total mass of the layer is zero, this is equal to $-\tfrac{1}{2}\rho\int g\zeta^2 d\sigma$.

It follows that

$$\delta E = \tfrac{1}{2}\rho^2 \iint \frac{\zeta_1\zeta_2 d\sigma_1 d\sigma_2}{D_{12}} - \tfrac{1}{2}\rho\int g\zeta^2 d\sigma.$$

The axes of the ellipsoid have been chosen so as to make our original E stationary, and the further condition to be satisfied is that δE shall be stationary.

Let us suppose that

$$\zeta = pe\,\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi),$$

which expression shall be deemed to include any one of the other types of harmonic.

Then it is shown in (51) of "Harmonics" that the potential of this layer at the surface of the ellipsoid is

$$\frac{3M}{k} e\,\mathfrak{P}_i^s(\nu_0)\,\mathfrak{Q}_i^s(\nu_0)\,\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi).$$

Since the mass of an element is $pe\rho\,\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)\,d\sigma$, we have

$$\tfrac{1}{2}\rho^2 \int \frac{\zeta_1\zeta_2 d\sigma_1 d\sigma_2}{D_{12}} = \tfrac{3}{2}\frac{M\rho}{k} e^2\,\mathfrak{P}_i^s(\nu_0)\,\mathfrak{Q}_i^s(\nu_0)\int [\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)]^2 p d\sigma.$$

With the value of g found in (36)

$$\tfrac{1}{2}\rho\int g\zeta^2 d\sigma = \tfrac{3}{2}\frac{M\rho}{k} e^2\,\mathsf{P}_1^1(\nu_0)\,\mathsf{Q}_1^1(\nu_0)\int [\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)]^2 p d\sigma.$$

Hence

$$\delta E = -\tfrac{3}{2}\frac{M\rho}{k}e^2\,\mathsf{P}_1{}^1(\nu_0)\,\mathsf{Q}_1{}^1(\nu_0)\left[1 - \frac{\mathfrak{P}_i{}^s(\nu_0)\,\mathfrak{Q}_i{}^s(\nu_0)}{\mathsf{P}_1{}^1(\nu_0)\,\mathsf{Q}_1{}^1(\nu_0)}\right]\int[\mathfrak{P}_i{}^s(\mu)\,\mathfrak{C}_i{}^s(\phi)]^2 p\,d\sigma.$$

In order that the new figure may be one of equilibrium, this expression must be stationary for variations of e. It follows that we must either have $e = 0$, which leads back to JACOBI's ellipsoid, or else

$$1 - \frac{\mathfrak{P}_i{}^s(\nu_0)\,\mathfrak{Q}_i{}^s(\nu_0)}{\mathsf{P}_1{}^1(\nu_0)\,\mathsf{Q}_1{}^1(\nu_0)} = 0.$$

This last condition is what M. POINCARÉ calls the vanishing of a coefficient of stability.* It shows that if ν_0 and β satisfy not only the condition for the Jacobian ellipsoid, namely, $\mathfrak{P}_2{}^1(\nu_0)\,\mathfrak{Q}_2{}^1(\nu_0) = \mathsf{P}_1{}^1(\nu_0)\,\mathsf{Q}_1{}^1(\nu_0)$, but also this equation, we have arrived at a figure which belongs at the same time to two series, and there is a bifurcation at this point. The form of the figure is found by attributing to e any arbitrary but small value.

§ 6. *The Properties of the Successive Coefficients of Stability.*

Corresponding to each harmonic deformation of the ellipsoid, there is a coefficient of stability of one of the two forms

$$1 - \frac{\mathfrak{P}_i{}^s(\nu_0)\,\mathfrak{Q}_i{}^s(\nu_0)}{\mathsf{P}_1{}^1(\nu_0)\,\mathsf{Q}_1{}^1(\nu_0)} \quad \text{or} \quad 1 - \frac{\mathsf{P}_i{}^s(\nu_0)\,\mathsf{Q}_i{}^s(\nu_0)}{\mathsf{P}_1{}^1(\nu_0)\,\mathsf{Q}_1{}^1(\nu_0)}.$$

These coefficients may be written $\mathfrak{K}_i^s$ or K_i^s according to an easily intelligible notation. The Jacobian ellipsoid is defined by ν_0, and the question arises as to the possibility of the vanishing of the several $\mathfrak{K}$'s as ν_0 gradually diminishes from infinity, that is to say, as the ellipsoid lengthens.

An harmonic of the first order merely denotes a shift of the centre of inertia along one of the three axes; one of the second order denotes a change of ellipticity of the ellipsoid. Since we must keep the centre of inertia at the origin, and since the ellipticity is determined by the consideration that the ellipsoid is a Jacobian, these harmonics need not be considered, and we may begin with those of the third order.

I shall not attempt to follow M. POINCARÉ in his masterly discussion of the properties of the coefficients of stability,† but will merely restate in my own notation the principal conclusions at which he has arrived.

* 'Acta Math.,' vol. 7, 1885, p. 321. The factors $\frac{1}{3}$ and $1/2n + 1$ (or $1/2i + 1$, if i is the degree of the harmonic) which occur in his form of the condition are included in my functions.

† Sections 10 and 12 of his memoir. I have to thank him for saving me from making a serious mistake in this portion of my work.

1st. The equation

$$\mathsf{P}_1^{\,1}(\nu)\,\mathsf{Q}_1^{\,1}(\nu) - \mathfrak{P}_i^{\,s}(\nu)\,\mathfrak{Q}_i^{\,s}(\nu) \text{ or } \mathsf{P}_i^{\,s}(\nu)\,\mathsf{Q}_i^{\,s}(\nu) = 0, \ (i > 2)$$

is not satisfied by any value of ν between 1 and infinity, if $\mathfrak{P}_i^s$ or P_i^s is divisible by $\left(\nu^2 - \frac{1+\beta^{\frac{1}{2}}}{1-\beta}\right)$. It appears from the forms of the functions as given in § 4 of "Harmonics" that the P functions are so divisible. These functions appertain to the types EES, OOC, OES, EOC, and therefore the ellipsoid cannot bifurcate into deformations of these types.

2nd. The equation has no solution if $\mathfrak{P}_i^s$ is divisible by $(\nu^2 - 1)^{\frac{1}{2}}$. We again see from § 4 of "Harmonics" that $\mathfrak{P}_i^s$ is so divisible if it is of the types OOS, EOS. Hence the ellipsoid cannot bifurcate into these types. The only types remaining are EEC, OEC.

3rd. The equation has no solution if any of the roots of $\mathfrak{P}_i^s(\nu) = 0$ lie outside the limits $+1$ to -1. The only $\mathfrak{P}_i^s$ of the types EEC, OEC which has all its roots inside the limits $+1$ to -1 is the zonal harmonic for which $s = 0$.

Hence the ellipsoid can only bifurcate into a zonal harmonic.

4th. The equation

$$\mathsf{P}_1^{\,1}\,\mathsf{Q}_1^{\,1} - \mathfrak{P}_i\,\mathfrak{Q}_i = 0 \quad (i > 2)$$

must have a solution between 1 and infinity for all values of i.

It follows from these four propositions that the Jacobian ellipsoid is stable for all deformations except the zonal ones, and that as it lengthens it must at successive stages bifurcate into each and all the zonal deformations.

5th. As the ellipsoid lengthens, the first coefficient of stability to vanish is that of the third zonal harmonic. This stage is the end of the stability of the Jacobian ellipsoids, and there is almost certainly exchange of stability with the pear-shaped figure defined by this harmonic.

6th. It has not been rigorously proved that there is only one solution of the equation $\mathfrak{K}_i = 0$ even in the case where $i = 3$, but M. POINCARÉ believes that this is almost certainly the case.

7th. The functions

$$\left.\begin{matrix}\mathfrak{P}_i^{\,s}(\nu_0)\\ \text{or}\\ \mathsf{P}_i^{\,s}(\nu_0)\end{matrix}\right\} \times \left.\begin{matrix}\mathfrak{P}_i^{\,t}(\nu)\\ \text{or}\\ \mathsf{P}_i^{\,t}(\nu)\end{matrix}\right\} - \left.\begin{matrix}\mathfrak{P}_i^{\,s}(\nu)\\ \text{or}\\ \mathsf{P}_i^{\,s}(\nu)\end{matrix}\right\} \times \left.\begin{matrix}\mathfrak{P}_i^{\,t}(\nu_0)\\ \text{or}\\ \mathsf{P}_i^{\,t}(\nu_0)\end{matrix}\right\}$$

have always the same sign as ν increases from ν_0 to infinity, provided that s and t are both greater than zero, and i greater than 2.

The seventh of the preceding propositions renders it easy to determine the relative magnitudes of all the $\mathfrak{K}$'s belonging to a single degree i.

In what follows I may take the symbols $\mathfrak{P}$, $\mathfrak{Q}$ as including also P, Q.

Now
$$\mathfrak{K}_i^s > = < \mathfrak{K}_i^t \quad \text{as,}$$

$$\mathfrak{P}_i^s(\nu_0)\,\mathfrak{Q}_i^s(\nu_0) - \mathfrak{P}_i^t(\nu_0)\,\mathfrak{Q}_i^t(\nu_0) < = > 0\,.$$

If we express the $\mathfrak{Q}$'s in terms of integrals this becomes

$$\int_{\nu_0}^{\infty} \frac{[\mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^t(\nu)]^2 - [\mathfrak{P}_i^s(\nu)\,\mathfrak{P}_i^t(\nu_0)]^2}{[\mathfrak{P}_i^s(\nu)\,\mathfrak{P}_i^t(\nu)]^2(\nu^2-1)^{\frac{1}{2}}\left(\nu^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}}\,d\nu < = > 0\,.$$

The seventh proposition shows that when s and t are greater than zero, and i is greater than 2, all the elements of the integral have the same sign. Hence the question is whether.

$$\frac{\mathfrak{P}_i^s(\nu_0)}{\mathfrak{P}_i^s(\nu)} < = > \frac{\mathfrak{P}_i^t(\nu_0)}{\mathfrak{P}_i^t(\nu)}\,.$$

Therefore we have to arrange all the $\dfrac{\mathfrak{P}_i^s(\nu_0)}{\mathfrak{P}_i^s(\nu)}$ in descending order of magnitude, and shall thereby obtain the non-zonal $\mathfrak{K}$'s in ascending order.

I wish first to show that these coefficients may to a great extent be sorted by considering the inequality.

$$\frac{\mathrm{P}_i^s(\nu_0)}{\mathrm{P}_i^s(\nu)} < = > \frac{\mathrm{P}_i^t(\nu_0)}{\mathrm{P}_i^s(\nu)} \quad (s = 1, 2, 3\ \ldots, i\,;\ t = 1, 2, 3\ \ldots, i).$$

Suppose, if possible, that whereas, for the ellipsoids defined by β, ν, ν_0,

$$\frac{\mathfrak{P}_i^s(\nu_0)}{\mathfrak{P}_i^s(\nu)} < \frac{\mathfrak{P}_i^t(\nu_0)}{\mathfrak{P}_i^t(\nu)}, \quad \text{yet} \quad \frac{\mathrm{P}_i^s(\nu_0)}{\mathrm{P}_i^s(\nu)} > \frac{\mathrm{P}_i^t(\nu_0)}{\mathrm{P}_i^t(\nu)}\,.$$

Then there must be some value of β for which

$$\mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^t(\nu) = \mathfrak{P}_i^s(\nu)\,\mathfrak{P}_i^t(\nu_0)$$

for all values of ν greater than ν_0.

It is almost obvious that there is no one value of β which renders this equation possible; but consider for example the case of $s = 2$, $t = 0$.

Now
$$\mathfrak{P}_3^2(\nu) = -\beta q_0 \mathrm{P}_3(\nu) + \mathrm{P}_3^2(\nu), \qquad \mathfrak{P}_3(\nu) = \mathrm{P}_3(\nu) + \beta q_2 \mathrm{P}_3^2(\nu_0).$$

It we substitute this in the equation we find

$$\mathrm{P}_3^2(\nu_0)\,\mathrm{P}_3(\nu) = \mathrm{P}_3^2(\nu)\,\mathrm{P}_3(\nu_0).$$

This can only be satisfied by $\nu = \nu_0$, and hence the hypothesis is negatived. Similarly the assumption of other values of s and t leads to an impossibility.

Thus we may consider the P functions in place of the $\mathfrak{P}$ functions.

Consider the inequality

$$\frac{P_i^s(\nu_0)}{P_i^s(\nu)} > = < \frac{P_i^{s+1}(\nu_0)}{P_i^{s+1}(\nu)}, \text{ for } s = 1, 2 \ldots, i-1.$$

If the inequality is determined for any value of ν, it is determined for all values. Now when ν is very large

$$P_i^s(\nu) = \frac{2i!}{2^i\, i!\, \overline{i-s}!}\nu^i, \qquad P_i^{s+1}(\nu) = \frac{2i!}{2^i\, i!\, \overline{i-s-1}!}\nu^i.$$

Hence our inequality becomes

$$(i-s)\, P_i^s(\nu_0) > = < P_i^{s+1}(\nu_0).$$

This inequality is of the same kind for all values of ν_0. Now $P_i^s(\nu_0)$ involves the factor $(\nu_0^2-1)^{\frac{1}{2}s}$ and $P_i^{s+1}(\nu_0)$ involves $(\nu_0^2-1)^{\frac{1}{2}(s+1)}$. Putting therefore $\nu_0^2 = 1+\epsilon$, the left-hand side involves $\epsilon^{\frac{1}{2}s}$ and the right $\epsilon^{\frac{1}{2}(s+1)}$. It follows that unless s is equal to i the left-hand side is greater than the right; but s is necessarily equal to $i-1$ at greatest.

Therefore
$$\frac{P_i^s(\nu_0)}{P_i^s(\nu)} > \frac{P_i^{s+1}(\nu_0)}{P_i^{s+1}(\nu)}.$$

Hence K's with smaller s are less than those with greater s.

It remains to discriminate between the two sorts of P-functions which occur in ellipsoidal harmonic analysis; that is to say we must determine

$$\frac{\mathfrak{P}_i^s(\nu_0)}{\mathfrak{P}_i^s(\nu)} > = < \frac{\mathsf{P}_i^s(\nu_0)}{\mathsf{P}_i^s(\nu)}.$$

Since the β of "Harmonics" is equal to $\dfrac{\kappa'^2}{2-\kappa'^2}$ in the present notation, when β and κ' are small we have by the formulæ of that paper

$$\mathfrak{P}_i^s(\nu) = P_i^s(\nu) + \tfrac{1}{2}\kappa'^2 q_{s+2} P_i^{s+2}(\nu) + \tfrac{1}{2}\kappa'^2 q_{s-2} P_i^{s-2}(\nu) + \ldots,$$

$$\mathsf{P}_i^s(\nu) = \frac{(\nu^2-1/\kappa^2)^{\frac{1}{2}}}{(\nu^2-1)^{\frac{1}{2}}}\left[P_i^s(\nu) + \frac{s+2}{2s}\kappa'^2 q_{s+2} P_i^{s+2}(\nu) + \frac{s-2}{2s}\kappa'^2 q_{s-2} P_i^{s-2}(\nu) + \ldots\right].$$

When ν is very great and κ' very small $\mathfrak{P}_i^s = \mathsf{P}_i^s$, so it suffices to determine the inequality

$$\mathfrak{P}_i^s(\nu_0) > = < \mathsf{P}_i^s(\nu_0);$$

and this may be considered for any value of ν_0 greater than unity. By taking ν_0 very large and κ' very small the inequality becomes

$$(\nu^2-1)^{\frac{1}{2}} > = < \left(\nu_0^2 - \frac{1}{\kappa^2}\right)^{\frac{1}{2}},$$

or
$$1 > = < \kappa.$$

But $\kappa < 1$, hence the first sign holds true and

$$\frac{\mathfrak{P}_i^s(\nu_0)}{\mathfrak{P}_i^s(\nu)} > \frac{\mathsf{P}_i^s(\nu_0)}{\mathsf{P}_i^s(\nu)},$$

whence $$\mathfrak{K}_i^s < \mathsf{K}_i^s.$$

Thus it follows that for order i

$$\mathfrak{K}_i^1 < \mathsf{K}_i^1 < \mathfrak{K}_i^2 < \mathsf{K}_i^2 \; . \; . \; . \; . \; < \mathfrak{K}_i^i < \mathsf{K}_i^i.$$

The order of magnitude of these coefficients is therefore completely determined.

As confirmatory of the correctness of this result it may be mentioned that I find that when $\gamma = 69° \; 50'$ and $\kappa = \sin 73° \; 56'$,

$$\mathfrak{K}_3^1 = \cdot 1765, \;\; \mathsf{K}_3^1 = \cdot 2990, \;\; \mathfrak{K}_3^2 = \cdot 4467, \;\; \mathsf{K}_3^2 = \cdot 4550, \;\; \mathfrak{K}_3^3 = \cdot 5719, \;\; \mathsf{K}_3^3 = \cdot 5876.$$

When $\gamma = 75°$ and $\kappa = \sin 81° \; 4'$ (another Jacobian ellipsoid) the numbers run ·130, ·224, ·460, ·465, ·604, ·614.

We see that for the harmonics of higher order the ellipsoid is more stable than it was and for those of lower order less stable.

§ 7. *The critical Jacobian Ellipsoid.*

From a number of preliminary calculations I saw reason to believe that the critical ellipsoid would be found within the region comprised between $\gamma = 69° \; 48'$ and $69° \; 50'$, and $\sin^{-1}\kappa = 73° \; 52'$ and $73° \; 56'$.

If we write

$$f(\gamma, \sin^{-1}\kappa) = \frac{\mathrm{E}}{\kappa'^2}\left(1 + \frac{\kappa^4 \sin^2\gamma \cos^2\gamma}{1 - \kappa^2 \sin^2\gamma}\right) - (2\mathrm{F} - \mathrm{E}) - \frac{\kappa^2 \sin\gamma \cos\gamma \,(1 + \kappa^2 \sin^2\gamma)}{\kappa'^2 (1 - \kappa^2 \sin^2\gamma)^{\frac{1}{2}}},$$

where the amplitudes of E and F are γ and their moduli κ, the existence of the Jacobian ellipsoid is determined by

$$f(\gamma, \sin^{-1}\kappa) = 0.^*$$

The coefficient of stability is

$$\mathfrak{K}_3(\gamma, \sin^{-1}\kappa) = 1 - \frac{\mathfrak{P}_3(\nu_0)\,\mathfrak{Q}_3(\nu_0)}{\mathsf{P}_1^1(\nu_0)\,\mathsf{Q}_1^1(\nu_0)}.$$

The formulæ for computing $\mathfrak{K}_3$ are given in § 4.

The values of E and F are from LEGENDRE's tables.

* See 'Roy. Soc. Proc.,' vol. 41, p. 323, where the formula is reduced to a form convenient for computation.

Now I find

$$f(69^\circ\ 48',\ 73^\circ\ 52') = +\ \cdot 000191\ ; \quad f(69^\circ\ 50',\ 73^\circ\ 52') = +\ \cdot 001319.$$
$$f(69^\circ\ 48',\ 73^\circ\ 56') = -\ \cdot 001186\ ; \quad f(69^\circ\ 50',\ 73^\circ\ 56') = -\ \cdot 000031.$$

$$\mathfrak{K}_3\,(69^\circ\ 48',\ 73^\circ\ 52') = +\ \cdot 001058\ ; \quad \mathfrak{K}_3\,(69^\circ\ 50',\ 73^\circ\ 52') = -\ \cdot 000885.$$
$$\mathfrak{K}_3\,(69^\circ\ 48',\ 73^\circ\ 56') = +\ \cdot 000655\ ; \quad \mathfrak{K}_3\,(69^\circ\ 50',\ 73^\circ\ 56') = -\ \cdot 000765.$$

By interpolation we get the following results :—

The Jacobian ellipsoid is given by

$$(\gamma - 69^\circ\ 48') - \cdot 59642\,(\sin^{-1}\kappa - 73^\circ\ 52') + \cdot 33091 = 0.$$

The vanishing of the coefficient of stability is given by

$$(\gamma - 69^\circ\ 48') + \cdot 041625\,(\sin^{-1}\kappa - 73^\circ\ 52') - 1\cdot 0890 = 0.$$

In these equations the minute of arc is the unit.

Solving them I find

$$\gamma = 69^\circ\ 48'\cdot 997 = 62^\circ\ 49'\cdot 0,$$

$$\sin^{-1}\kappa = 73^\circ\ 54'\cdot 225 = 73^\circ\ 54'\cdot 2.$$

With these values I find that the three axes a, b, c, where $abc = \mathrm{a}^3$ are

$$\frac{a}{\mathrm{a}} = \cdot 650659,$$

$$\frac{b}{\mathrm{a}} = \cdot 814975,$$

$$\frac{c}{\mathrm{a}} = 1\cdot 885827.$$

The last place of decimals in these is certainly doubtful.

The formula for ω^2 is given in (35).

Now $\quad \Psi = \dfrac{2}{k}\,\mathfrak{P}_0(\nu_0)\,\mathfrak{Q}_0(\nu_0), \quad k = c\kappa\sin\gamma, \quad \mathfrak{P}_0(\nu_0)\,\mathfrak{Q}_0(\nu_0) = \kappa \mathrm{F}.$

Then since $a = c\cos\gamma$, $b = c\Delta$,

$$\frac{\omega^2}{2\pi\rho} = \frac{2\mathrm{F}\Delta\cot\gamma - \frac{6}{1+\Delta^{-2}+\sec^2\gamma}}{1+\Delta^2 - \frac{6}{1+\Delta^{-2}+\sec^2\gamma}}.$$

In this formula, F, γ, Δ must correspond with values interpolated amongst those used in obtaining the solution.

From this I find

$$\frac{\omega^2}{2\pi\rho} = \cdot 1419990 = \cdot 14200.$$

In the paper on the Jacobian ellipsoid referred to above the moment of momentum is tabulated by means of μ, where the moment of momentum is $(\frac{4}{3}\pi\rho)^{\frac{3}{2}}\mathrm{a}^5\mu$. The formula for μ is given in (25) of that paper, and, modified to suit the present notation, is

$$\mu = \frac{3^{\frac{1}{2}}}{5}(\Delta\cos\gamma)^{-\frac{2}{3}}(1+\Delta^2)\left(\frac{\omega^2}{4\pi\rho}\right)^{\frac{1}{2}}.$$

For the critical ellipsoid I find $\mu = \cdot 389570$.

The following table gives the numerical values for a number of Jacobian ellipsoids, beginning with the initial one and terminating just beyond instability. The last line gives the corresponding values for the critical ellipsoid.

JACOBI'S Ellipsoids.*

γ.	$\sin^{-1}\kappa$.	$\cos^{-1}\Delta$.	a/a.	b/a.	c/a.	$\omega^2/2\pi\rho$.	μ.
° ′ ″	° ′	° ′					
54 21 27 . .	0 0	0 0	·6977	1·1972	1·1972	·18712	·30375
55	$17\frac{3}{4}$	$14\frac{1}{4}$	·697	1·179	1·216	·18706	·304
57	$34\frac{3}{4}$	$28\frac{1}{2}$	·696	1·123	1·279	·186	·306
60	49 7	40 54	·6916	1·0454	1·3831	·1812	·3134
65	64 19	54 46	·6765	·9235	1·6007	·1659	·3407
70	74 12	64 43	·6494	·8111	1·899	·1409	·3920
69 49 . .	73 54	64 24	65066	·81498	1·88583	·14200	·38957

* I have been criticised with respect to my paper on JACOBI'S ellipsoid, from which these results are extracted, by M. S. KRÜGER (Nieuw Archief voor Wiskunde, Tweede Reeks, Derde Deel and 'Ellipsoidale Evenwichtsvormen,' &c., Thesis for Degree of Doctor, Leiden, J. W. van Leeuwen, Hoogewoerd 89, 1896), because I wrote it in ignorance of certain previous work, especially of a paper by PLANA ('Ast. Nachr.,' 36, n. 851, c. 169). But I cannot but congratulate myself on my ignorance, since it appears that PLANA gave a number of numerical results which were wholly wrong. A knowledge of that paper would no doubt have caused me much further trouble.

My paper gives a number of solutions of the problem which I believe to be correct. Unfortunately the methods of the paper are clumsy, and there are several mistakes. The formula for ω^2 used in this present paper, is much better than that given there.

The complicated formula on p. 325 is susceptible of reduction to a simple form, for on substituting for γ its approximate form (i) we have simply

$$\gamma - \delta = \tfrac{1}{4}\kappa^2 \sin\delta\cos\delta,$$

where $$\delta = 54^\circ\,21'\,27''.$$

The final numerical result was, however, nearly right, for I now find

$$\sin^2\alpha = 10^{\cdot 9266821}\sin(\gamma-\delta),$$

whereas I had ·9266528. The $\sin\alpha$ is the same as the κ used here.

The formula at the top of p. 326 which is reproduced as (22) on p. 828 is, I think, illusory, for if in the

In order to determine the question as to whether or not it is possible that $\mathfrak{K}_3 = 0$ should have another solution than that found in the next section, I have computed the value of this coefficient for the Jacobian ellipsoid $\gamma = 75°$, $\kappa = \sin 81° \; 4'{\cdot}4$, and find it to be $-6{\cdot}627$. From the manner in which the numbers in the computation present themselves, it is obvious that for more elongated ellipsoids $\mathfrak{K}_3$ will always remain negative, and will become numerically greater. I have therefore not thought it necessary to seek for an algebraic proof that there is no second root of the equation.

Very long Jacobian ellipsoids tend to become figures of revolution, and the coefficients of stability tend to assume the forms

$$1 - \frac{P_i(\nu)\,Q_i(\nu)}{P_1^{\,1}(\nu)\,Q_1^{\,1}(\nu)}.$$

The forms of these functions are well known, and I think that fair approximations to the incidences of the successive figures of bifurcation might be derived from the vanishing of this expression.

For example

$$P_1^{\,1}(\nu)\,Q_1^{\,1}(\nu) = \tfrac{1}{2}\left[\nu - (\nu^2 - 1)\log\left(\frac{\nu+1}{\nu-1}\right)^{\frac{1}{2}}\right]$$

$$P_4(\nu)\,Q_4(\nu) = \tfrac{1}{64}\left[(35\nu^4 - 30\nu^2 + 3)\log\left(\frac{\nu+1}{\nu-1}\right)^{\frac{1}{2}} - \tfrac{5}{3}\nu\,(21\nu^2 - 11)\,(35\nu^4 - 30\nu^2 + 3)\right].$$

I have not, however, attempted to solve the equation found by equating these two expressions to one another.

Even when $i = 3$ and $\gamma = 69° \; 49'$ (the critical Jacobian) this rough approximation makes the coefficient of stability very small, but it is to be admitted that $P_1^{\,1}\,Q_1^{\,1}$ and $P_3 Q_3$ differ very sensibly from $\mathbf{P}_1^{\,1}(\nu)\,\mathbf{Q}_1^{\,1}(\nu)$ and $\mathfrak{P}_3(\nu)\,\mathfrak{Q}_3(\nu)$, although in such a way that the errors compensate one another.

first term we put $\gamma = \delta + \frac{1}{4}\kappa^2 \sin\delta\cos\delta$ (as is clearly allowable in approximation) the term with coefficient κ^2 or $\sin^2\alpha$ disappears. This shows that it was necessary to proceed in the approximation as far as κ^4 in order to obtain a result.

The methods of approximation adopted on pp. 326–7 are of doubtful propriety, but will, I think, lead to nearly correct results. There is, however, a mistake towards the bottom of p. 327 which runs on to the end. M. KRÜGER correctly points out that the second line of formula (24) p. 329 should run

$$\tfrac{1}{2}\cos^2\alpha\left[\frac{1}{\sin\gamma}\log_e\cot\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\gamma\right).\left(\tfrac{15}{8} + 3\tan^2\gamma + \tan^4\gamma\right) - \tfrac{15}{8} - \tfrac{29}{8}\tan^2\gamma - \tfrac{7}{4}\tan^4\gamma\right].$$

Lastly, on p. 335, line 13, for $C = 0{\cdot}3573$, read $C = 0{\cdot}5379$; and on p. 336, line 7, for $1{\cdot}3573$, read $1{\cdot}5379$; and for $\frac{\mathfrak{a}}{a} = 1{\cdot}696$, read $\frac{a}{\mathfrak{a}} = 4{\cdot}65$.

§ 8. *The pear-shaped Figure of Equilibrium.*

By (21) the normal displacement δn for the third zonal harmonic deformation may be written

$$\delta n = e \frac{z\left[q'^2 z^2 - q^2 x^2 - (3 - 4q^2) y^2 - c^2 q^2 q'^2 \sin^2\gamma\right]}{c^2 q'^2 (1 - q^2 \sin^2\gamma)\,(x^2/\cos^4\gamma + y^2/\Delta^4 + z^2)^{\frac{1}{2}}},$$

subject to the condition

$$\frac{x^2}{\cos^2\gamma} + \frac{y^2}{\Delta^2} + z^2 = c^2.$$

The expression has been arranged so that when $x = y = 0$, $z = c$, we have $\delta n = e$. Hence $+e$ and $-e$ are the normal displacements at the stalk and blunt end of the pear respectively.

In the section $y = 0$, this may be written

$$\delta n = \frac{e\cos\gamma}{q'^2} \cdot \frac{z\,(z^2 - c^2 q^2)}{c^2\,(c^2 - z^2\sin^2\gamma)^{\frac{1}{2}}}.$$

The nodal points are given by $\frac{z}{c} = \pm q = \pm \cdot 758056$.

In the section $x = 0$, since $\kappa^2 = q^2 \frac{4 - 5q^2}{3 - 4q^2}$, it may be written

$$\delta n = e\frac{\Delta\,(4 - 5q^2)}{q'^2} \cdot \frac{z\,(\kappa^2 z^2 - c^2 q^2)}{c^2\kappa^2\,(c^2 - \kappa^2 z^2 \sin^2\gamma)^{\frac{1}{2}}}.$$

The nodal points are given by $\frac{z}{c} = \pm\frac{q}{\kappa} = \pm \cdot 788986$.

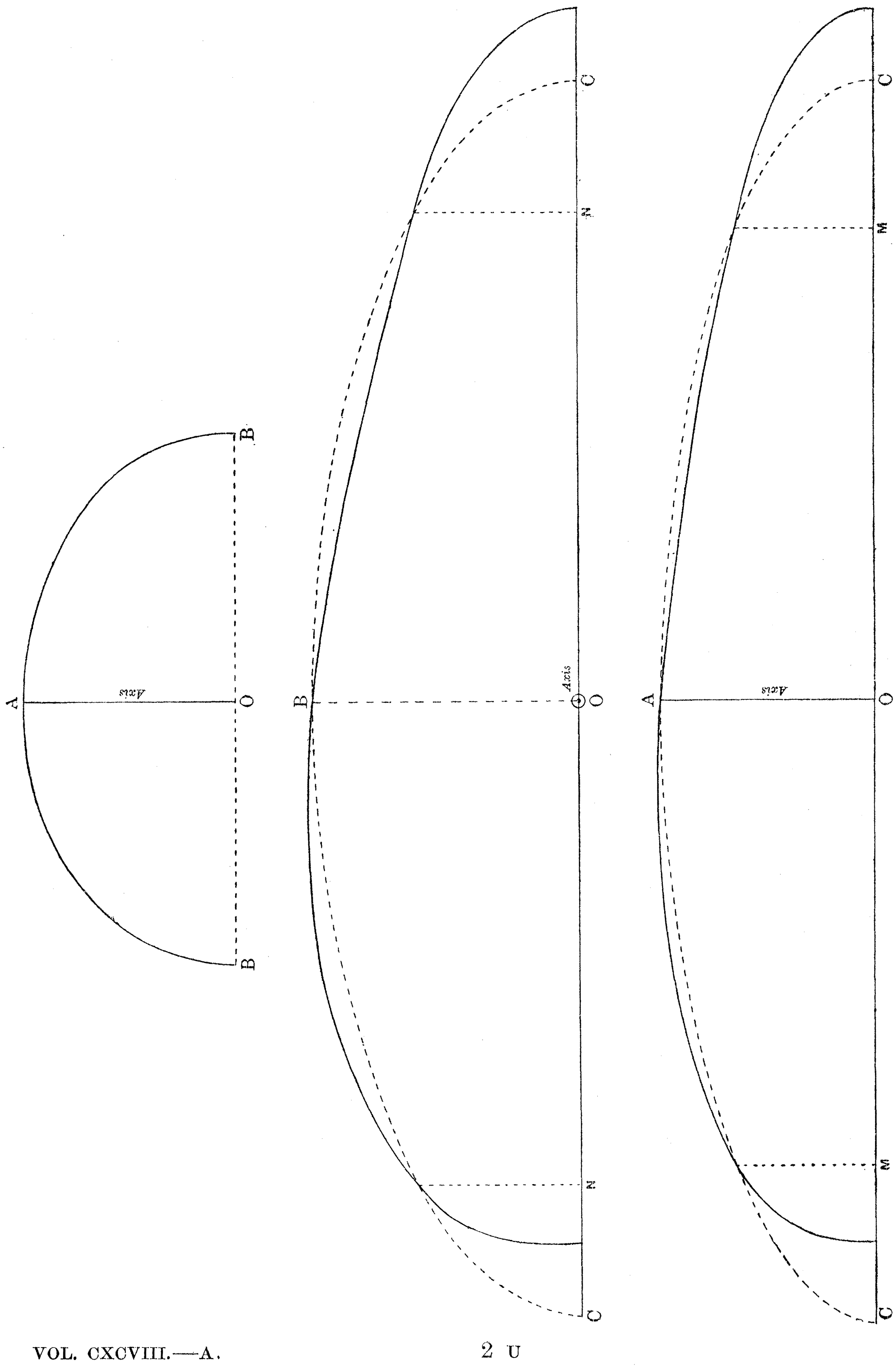
The section $z = 0$ is obviously another nodal line for all sections.

By means of these formulæ it is easy to compute the normal displacements from the surface of the critical Jacobian.

The figure opposite showing the three sections $x = 0$, $y = 0$, $z = 0$, is drawn from these formulæ, the dotted line being the critical Jacobian and the firm line the pear. The scale of the normal displacements is, of course, arbitrary.

Comparison with M. POINCARÉ's sketch shows that the figure is considerably longer than he supposed.

In this first approximation the positions of the nodal lines are independent of the magnitude of e, and they lie so near the ends that it is impossible to construct an exaggerated figure, for if we do so the blunt end acquires a dimple, which is absurd. It might have been hoped that such an exaggeration would afford us some idea of the mode of development of the pear.



PEAR-SHAPED FIGURE OF EQUILIBRIUM.

OA = ·65066, OB = ·81498, OC = 1·88583; $\frac{\omega^2}{2\pi\rho} = \cdot 14200$, $\frac{OM}{OC} = \cdot 75806$, $\frac{ON}{OC} = \cdot 78899$.

M. SCHWARZSCHILD has remarked* that it is not absolutely certain that the principle of exchange of stability holds with reference to this figure, and that we cannot feel absolutely certain that the pear is stable unless we can prove that the moment of momentum is greater than in the critical Jacobian.

With reference to this objection, M. POINCARÉ writes to me as follows :—

"Faisons croître le moment de rotation, que j'appellerai M. Deux hypothèses sont possibles.

"Ou bien pour $M < M_0$ (the moment of momentum of the Jacobian), nous aurons une seule figure, *stable*, à savoir l'ellipsoïde de JACOBI, et pour $M > M_0$ trois figures, une instable, l'ellipsoïde, et deux stables (d'ailleurs égales entre elles), les deux figures pyriformes.

"Ou bien pour $M < M_0$, nous aurons trois figures d'équilibre, deux pyriformes instables, une stable, l'ellipsoïde, et pour $M < M_0$ une seule figure instable, l'ellipsoïde—auquel cas la masse fluide devrait se dissoudre par un cataclysme subit.

"Il y a donc à vérifier si pour les figures pyriformes, $M >$ ou $< M_0$."

It seems very improbable that the latter can be the case; but this opinion is not a proof.

Since ω^2 is stationary for the initial pear, a small change in the angular velocity will certainly produce a great change in the figure of the pear. If this investigation has, in fact, its counterpart in the genesis of satellites and planets, it seems clear that the birth of a new body, although not cataclysmal, is rapid.

§ 9. *Summary.*

It is possible by the methods explained in my previous paper on "Harmonics" to form rigorous expressions for the ellipsoidal harmonics of the third degree. Accordingly in § 1 I proceed to form those functions. In § 2 the notation is changed with a view to convenience in subsequent work, and for the sake of completeness the harmonics of the first and second degrees are also given. In § 3 the corresponding solid harmonics are expressed in rectangular co-ordinates x, y, z. In § 4 I find the Q-functions, the harmonic functions of the second kind, and express the results in terms of the elliptic integrals E and F. It appears that both the P- and Q-functions of the third degree of harmonics occur in three pairs which have the same algebraic forms, and that in each pair one of them only differs from the other in the value of a certain parameter. There is, lastly, a seventh function which stands by itself; this last corresponds to the solid harmonic xyz.

In § 5 the equations for JACOBI's ellipsoid are determined by the consideration that the energy must be stationary, and the superficial value of gravity is found in terms of the appropriate P- and Q-functions. I then proceed to find the additional terms

* "Die Poincarésche Theorie des Gleichgewichts," 'Annalen der K. Sternwarte, München,' Bd. III.

in the energy when the mass of fluid is subject to an ellipsoidal harmonic deformation. This section is a paraphrase of M. POINCARÉ's work, but the notation and manner of presentation are somewhat different. The additional terms in the energy are shown to involve a certain coefficient, which is called by M. POINCARÉ a coefficient of stability. It is clear that when any coefficient vanishes we are at a point of bifurcation, and the particular Jacobian ellipsoid for which it vanishes is also a member of another series of figures of equilibrium.

In § 6 the principal properties of these coefficients, as established by M. POINCARÉ, are enumerated. He has shown that the ellipsoid can bifurcate only into figures defined by zonal harmonics; that it must do so for all degrees, and that the first bifurcation occurs with the third zonal harmonic. The order of magnitude of the coefficients of the several orders and of the same degree is determined. A numerical result seems to indicate that as the ellipsoid lengthens, it becomes more stable as regards deformations of the third degree and of higher orders, and less stable as regards the lower orders of the same degree.

In § 7 the numerical solution of the vanishing of the coefficient corresponding to the third zonal harmonic is found, and it is shown that the critical ellipsoid has its three axes proportional to ·65066, ·81498, 1·88583, and that the square of the angular velocity is given by $\frac{\omega^2}{2\pi\rho} = \cdot 14200$. A short table is also given showing the march of the axes of the Jacobian ellipsoids from their beginning on to instability at this critical stage. The nature of the formula for the third zonal coefficient of stability seems to show that it can only vanish once—a point which it appears that M. POINCARÉ found himself unable to prove rigorously.

A suggestion is made for the approximate determination of the bifurcations into the successive zonal deformations, but no numerical results are given.

In § 8 the nature of the pear-shaped figure is determined numerically, and the reader may refer to the figure above, where it is delineated. It will be seen to be longer than was shown in M. POINCARÉ's conjectural sketch.

If, as M. POINCARÉ suggests, the bifurcation into the pear-shaped body leads onward stably and continuously to a planet attended by a satellite, the bifurcation into the fourth zonal harmonic probably leads unstably to a planet with a satellite on each side, that into the fifth to a planet with two satellites on one side and one on the other, and so on.

The pear-shaped bodies are almost certainly stable, but a rigorous and conclusive proof is wanting until the angular velocity and moment of momentum corresponding to a given pear are determined. To do this further approximation is needed.